Journal of Engineering Mathematics, Vol. 13, No. 1, January 1979

Wave trapping with shore absorption

R. E. MEYER and J. F. PAINTER

Mathematics Department, University of Wisconsin, Madison WI 53706, USA

(Received June 13, 1978)

SUMMARY

Surface waves trapped over naturally gentle seabed topographies near coasts may decay in time due to energy absorption at the shore. The decay rate is computed in terms of the shore reflection coefficient for round islands and for seabed depth varying only normally to a straight beach, as in laboratory channels. This decay rate is liable to be considerably larger than that due to energy leakage to the open sea, but is still relatively small, since it is proportional to (frequency)$^{-1}$. The resonant response of such modes may therefore still be important.

1. Introduction

Ursell [1] first showed that surface waves can be trapped near coasts even in an unbounded ocean. For periods of less than a few hours, such trapping results from the influence of the seabed: waves over deeper water travel faster, and the crests and troughs therefore turn gradually towards the shallows. The reasonance that might arise from such trapping were studied by Longuet-Higgins [2] and by Shen et al. [3] for axisymmetric seabeds and for seabeds with depth dependent on only one Cartesian coordinate, as in typical laboratory channels (for brevity, referred to as channel topograhies hereafter). Longuet-Higgins' investigation was based on the longwave approximation away from shores with emphasis on certain, very special topographies, for which he discovered strong, narrow resonances in an unexpected frequency range. The study of Shen et al. included shores and covered all but the longest waves on an approximation based on the natural gentleness of sedimentary seabed topographies, for which the seabed slope, curvature, etc., are characterized by a small parameter. Large numbers of potentially resonant frequencies were found for general topographies by the spectral method of Keller's ray theory ('Geometrical Optics Approximation').

Both investigations showed trapping to be complete and resonance, of classical nature, for channel topographies. But for axisymmetric topographies, both found trapping to be necessarily incomplete, with energy leakage to the open sea, whence resonance must be of a non-classical nature (referred to as quasi-resonance or spectral concentration in quantum theory). Since the same must be anticipated for real ocean topographies, the axisymmetric case aroused much interest as the simplest topography realistic in this important respect.

The results of Shen et al. [3] demonstrated the power and flexibility of rational ray theory, but its basic formulation [4] is designed for self-adjoint problems and is unsuited to a discussion of leakage or spectral concentration. To obtain information on the degree of resonant response or measure of spectral concentration, a more broadly based theory of surface wave refraction is needed.

The gentleness of sedimentary seabeds implies that, over a distance of but a few wavelengths, even long surface waves travel virtually unchanged. Over longer distances, however, they are gradually modulated, and over larger continental shelves, even waves of tsunami length tend to be modified very substantially. Such modulation is characterized by a small parameter ϵ representing the ratio of the wavelength scale to the geographical scale of the distances over which the water depth changes by a significant fraction. A general theory aiming to predict the effects of such modulation on classical, small-amplitude waves to the first relevant approximation with respect to ϵ will here be referred to as refraction theory.

Berkhoff [5] and Lozano and Meyer [6] have proposed that such a theory can be based on a refraction equation (Section 2) of Helmholtz type. It yields the appropriate approximation to the known exact solutions of the classical linear surface wave equations, it yields the rational ray theory [6] and it has been derived rigorously from the classical, linear equations under restricted circumstances [7]. Accordingly, it has been adopted as the basis of the present investigation.

Lozano and Meyer [6] have used it to predict leakage and response for trapped wave modes of axisymmetric islands of general shape and have found large numbers of frequencies of extraordinarily small leakage and correspondingly large, narrow resonant response. Indeed, the response is *exponentially large* in ϵ, and an unusually sophisticated analysis extending mathematical WKB theory was required for its treatment.

All these investigations, however, if they involved shores, were based on the assumption of solutions bounded at the shore, which implies perfect reflection of energy from the shore. One of the main reasons for this was the lack of concrete information on the shore reflection. It should be emphasized here that the study of shore reflection is not within the scope of refraction theory, however small the beach slope; sufficiently close to shore, the mechanism of surface waves is not at all one of modulation in the sense just described. For refraction theory, the shore reflection coefficient therefore represents a boundary condition that needs to be specified from external sources of information. Those sources, unfortunately, appear to be largely lacking at the time of writing.

For axisymmetric islands, however, the theory of the refraction equation has been advanced by Lozano and Meyer [6] to a stage where it becomes possible to predict the dependence of the eigenvalues on the shore reflection coefficient analytically in a general way, and this is the object of the following. Similar information for channel topographies may be of value for experimental studies of wave trapping and might even open an avenue for obtaining, in turn, information on reflection coefficients from measurements of edge wave decay.

For channel topographies, trapping is complete and trapped modes do not decay in time, if shore reflection is perfect and viscous dissipation, negligible (as it is on the field scale to a substantial degree [2]). Up to this point, therefore, Keller's rational ray theory [3] has been sufficient for channel topographies. Shore absorption, however, introduces a time decay for edge waves, and it becomes necessary to develop the refraction theory for channel topographies also on the more general basis of the refraction equation. This is done in Sections 3, 4. The analogous theory for round islands is then briefly summarized in Section 5.

The main result is that the logarithmic time decrement of the trapped wave modes due to shore absorption is

$$Im\, \omega \sim (4\omega\, |\xi_c'(1)|)^{-1} \log |R_A|$$

to the first approximation in ϵ where ω denotes the frequency (predicted by Shen et al. [3]), R_A denotes the amplitude reflection coefficient (so that $1 - |R_A|^2$ is the fraction of wave energy absorbed), and $|\xi_c'(1)|$ denotes a certain WKB integral. This decay formula applies to both channel topographies and round islands; of course the frequencies differ in the two cases, and so does the WKB integral, which is defined for channel topographies by (22) below and for round islands, by (37).

The strongly resonant frequencies are relatively large, of the order of the reciprocal square root of the seabed slope, and the decay rate is correspondingly small. Of course, it is much larger than for round inslands with perfect shore reflection, where it is exponentially small in the seabed slope. Accordingly, the peak of the resonant response of the trapped modes to excitation by waves of the same frequency incident from the open sea [2, 6, 8] is much reduced by any degree of shore absorption. On the other hand, the frequency bandwidth of the response is correspondingly broadened. Since the peak response remains a relatively large one, it is still to be anticipated that these high-frequency modes make a prominent contribution to the total wave spectrum in many circumstances.

2. Refraction equation

When the mean water depth h does not change significantly over the distance of a wavelength, it is plausible that the local structure of the motion depends, to a first approximation, on the local depth $h(x,y)$ but not on its gradient or higher derivatives. For small-amplitude waves governed by the classical, linear theory of surface waves [9] the first approximation to the velocity potential of waves of period $2\pi/\omega$ must then be of the form

$$\phi(x, y, z, t) = \exp(-i\omega t) \cosh [k(z + h)]\, \Psi(x, y)/\cosh (kh)$$

where Ψ is the surface value of the potential and the cosh factor represents the classical, vertical structure of waves over water of constant depth h. To satisfy the surface boundary condition, the wave number function $k(x, y)$ must be related to the depth $h(x, y)$ by the classical dispersion relation

$$k \tanh (kh) = \epsilon\omega^2 \equiv \eta \tag{1}$$

The factor ϵ appears because h and the vertical coordinate z are naturally measured in units of a typical depth H, but x, y are naturally measured in units of a topographic scale $H/\epsilon \gg H$ and it is convenient to adopt that scale also for $1/k$; η serves to abbreviate the scaled, square frequency $\epsilon\omega^2$. Use of the full dispersion relation (1) will permit impartial coverage of all wave lengths.

Substitution into the classical, linear surface wave equations

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \epsilon^{-2}\partial^2\phi/\partial z^2 = 0 \quad \text{for} \quad 0 > z > -h(x,y),$$
$$\partial\phi/\partial z = -\epsilon\, \partial^2\phi/\partial t^2 \quad \text{at} \quad z = 0,$$
$$(\partial\phi/\partial x)\, \partial h/\partial x + (\partial\phi/\partial y)\partial h/\partial y + \epsilon^{-2}\partial\phi/\partial z = 0 \quad \text{at} \quad z = -h,$$

and vertical integration lead to an equation for $\Psi(x, y)$ that can be plausibly approximated [6] by the refraction equation

$$\nabla(G\,\nabla\,\Psi) + k^2\epsilon^{-2}G\,\Psi = 0 \tag{2}$$

where the wave depth function

$$G = [\sinh(2kh) + 2kh]/[4\,k\cosh^2(kh)] \tag{3}$$

can be interpreted as a measure of the degree to which the surface wave feels the seabed; in the longwave limit $kh \to 0$, (1) gives $k^2 \sim \eta/h$ and $G \sim h$. For $\psi(x, y) = G^{\frac{1}{2}}\Psi$, the refraction equation becomes

$$\nabla^2\psi + [\epsilon^{-2}k^2 - G^{-\frac{1}{2}}\nabla^2(G^{\frac{1}{2}})]\,\psi = 0. \tag{4}$$

The questions arising in connection with wave trapping are wave scattering problems defined by three pieces of information, namely the differential equation (4), a shore condition specifying the local relation between the incident and reflected waves there, and a radiation condition specifying the role of the open sea (or the wave maker). The main function of mathematics is to establish the characteristic relation between the shore and radiation conditions brought about by the differential equation describing the physical wave mechanism. That relation involves only certain average properties of the seabed topography, which therefore needs to be specified only in general terms.

For ease of presentation, this will be done separately for channel topographes (Section 3) and round islands (Section 5). The rigorous WKB analysis furnishing the first asymptotic approximation to the characteristic relation for channel topographies will be sketched in Section 3. For the similar analysis for round islands, reference may be made to [6] and it will be summarized only briefly in Section 5.

3. Channel topography

When the depth depends only on one Cartesian coordinate x, the same follows for k and G, by (1) and (3), and the potential may be Fourier-analyzed in the y-direction in terms of modes

$$\Psi(x, y) = w(x)e^{iny}$$

with integer n, for which the refraction equation (4) reduces to

$$d^2w/dx^2 = [n^2 - k^2/\epsilon^2 + G^{-\frac{1}{2}} d^2 G^{\frac{1}{2}}/dx^2]\, w \tag{5}$$

and for sufficiently small ϵ, this square bracket is close to $\epsilon^{-2} f(x)$ with

$$f(x) = n^2\epsilon^2 - [k(x)]^2, \tag{6}$$

except right at a shore, where $G^{-\frac{1}{2}}(G^{\frac{1}{2}})''$ is singular.

The depth $h(x)$ will be assumed to increase monotonely with x, with typical shore character at $x = 0$ so that $h'(0) > 0$. Then the dispersion relation (1) shows the wave number $k(x)$ to be singular at $x = 0$ and for real frequency, monotone decreasing with x to a limit $k_\infty \geqslant \eta$ as $x \to \infty$. Only values of $|n|\epsilon$ above the cut-off value k_∞ will be considered here, and the function $f(x)$ in (6) then has a root X. For simplicity, extreme shelves [3] will be excluded by the assumption that $f(x)$ has only one real root for real η. For natural seabeds, $h(x)$ is smooth, and since only certain average properties are relevant, no generality is lost in assuming $h(x)$ analytic on a neighborhood of $[0, \infty)$ (including, at large $|x|$, a sector $|\arg x| \leqslant \alpha_0$ with ϵ-independent $\alpha_0 > 0$).

The wave depth $G(x)$ is then similarly analytic, with simple root at $x = 0$, and $f(x)$ is analytic on a neighborhood of $(0, \infty)$. By (1), the wave number k is also analytic in $\eta = \epsilon\omega^2$, for fixed x, on a neighborhood N of $\eta = 1$, and this also carries to f and $G^{\frac{1}{2}}$. For near-real η, the shore singularity of f remains at $x = 0$ and the root $X(\eta)$ of f remains near-real and simple.

It follows that $x = X(\eta)$ is a simple turning point [10] of (5) and $x = 0$, a singular turning point of order -1. Their Stokes lines (more precisely, anti-Stokes or principal lines) are the curves issuing from a turning point on which the WKB variable (or Liouville-Green or Langer variable)

$$\int^x [f(s)]^{\frac{1}{2}} ds$$

has constant real part. Standard theory [10 or 11] shows one Stokes line L_0 to issue from $x = 0$ and three, L_1, L_2 and L_*, from $x = X(\eta)$ (Fig. 1). They do not intersect [6], except that L_0 and L_* both coincide for real η with the segment $(0, X)$ of the real x-axis.

For definiteness, define

$$\xi_0(x) = \int_0^x [f_0(s)]^{\frac{1}{2}} ds, \tag{7}$$

$$\xi_i(x) = \int_X^x [f_i(s)]^{\frac{1}{2}} ds, \quad i = 1,2,*,$$

with branch $f_i^{\frac{1}{2}}$ of $f^{\frac{1}{2}}$ that makes $\arg \xi_i(x) = \pi/2$ and $|\xi_i(x)|$ increasing with $|x|$ and $|x - X|$, respectively, on L_i for $i = 0,1,2,*$. For consistency with [6] the choice may be made for $\eta = 1$ so that $\arg f'(X) = 2\pi$ and $\arg(x - X) = -\pi$ for $0 < x < X$ and

$$\arg f_0 \to \pi, \qquad \arg \xi_0 \to \pi/2 \qquad \text{on } L_0 \text{ as } x \to 0, \tag{8}$$

$$\arg \frac{f_i(x)}{x-X} \to \begin{cases} 4\pi \\ 2\pi, \\ 0 \end{cases} \quad \arg \frac{\xi_i(x)}{(x-X)^{\frac{3}{2}}} \to \begin{cases} 2\pi \\ \pi, \\ 0 \end{cases} \quad \arg(x-X) \to \begin{cases} -\pi \\ -\pi/3 \\ \pi/3 \end{cases}$$

$$\text{on } \begin{cases} L_* \\ L_1 \\ L_2 \end{cases} \tag{9}$$

as $|x-X| \to 0$. For $\eta \in N$, it is then defined by continuity in η. This assures pairwise overlap between suitable neighborhoods D_i of the respective Stokes lines (Fig. 1) which can be envisaged first as discs about $\xi_i = 0$ cut along the negative imaginary ξ_i-axis; the Stokes line L_i coincides with the positive imaginary axis (Fig. 2).

If the domains D_i are restricted so that their images Δ_i in the x-plane (Fig. 1) remain within the domain of analyticity of $f(x)$, standard theory [12 or 13] assures on each D_i an exact solution pair $u_i^{\pm}(x)$ of (5) with asymptotic approximation

$$\begin{aligned} u_i^{\pm} &\sim f_i(x)^{-\frac{1}{4}} \exp[\pm \xi_i(x)/\epsilon]\{c_i + O(\epsilon)\}, \\ du_i^{\pm}/dx &\sim \pm \epsilon^{-1} f_i^{\frac{1}{4}} \exp[\pm \xi_i/\epsilon]\{c_i + O(\epsilon)\} \end{aligned} \tag{10}$$

on a subdomain D_i'. The latter is obtained by deleting from D_i an arbitrarily small, but ϵ-independent, neighborhood of $\xi_i = 0$ and an equally wide strip about the negative imaginary ξ_i-axis (Fig. 2). The normalization constants c_i are traditionally defined by $|c_i| = 1$ and $4 \arg c_i = \lim \arg f_i(x)$ as $x \to X$ on L_i (for $i = 1,2,*$, and as $x \to 0$ for $i = 0$). These are the traditional WKB solutions whose exponential factor is purely oscillatory on L_i. With the choice $Re\omega > 0$ in (1), u_i^+ represents a wave travelling away from X on L_i (for $i = 1,2,*$, and from 0, for $i = 0$), while u_i^- represents a wave incident on the turning point along L_i.

It follows that u_i^+ and u_i^- are independent solutions of (5) on D_i and must, on $D_i \cap D_j$, each be linear combinations of u_j^+ and u_j^- with x-independent coefficients. Standard theory [12] gives these relations as

$$u_*^+(x) = \mu\, u_0^-(x), \qquad u_*^-(x) = \mu' u_0^+(x) \tag{11}$$

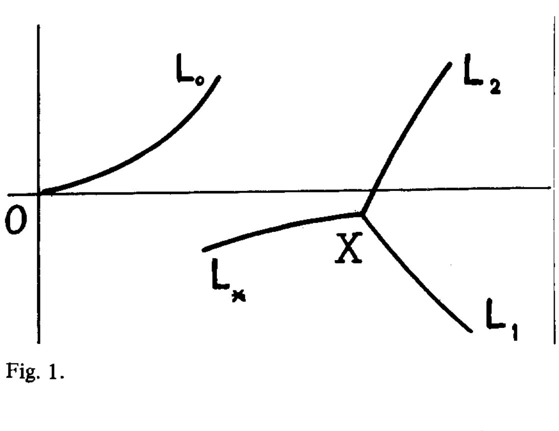

Fig. 1.

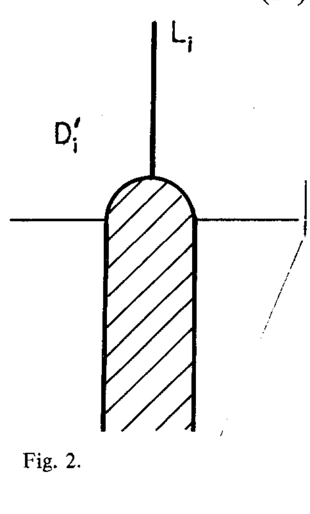

Fig. 2.

with

$$\mu'/\mu = \exp\left[-2\xi_c(\eta)/\epsilon\right]\{1 + O(\epsilon)\},$$
$$\xi_c(\eta) = \int_0^X [f_0(x)]^{\frac{1}{2}}\, dx \tag{12}$$

and $\xi_c = i\,|\,\xi_c\,|$ for real η, by (8), and

$$u_1^+(x)e^{-i\pi/6} = u_*^-(x)\{1 + O(\epsilon)\} - iu_*^+(x)\{1 + O(\epsilon)\},$$
$$u_1^-(x)e^{-i\pi/6} = u_*^+(x). \tag{13}$$

The solution $w(x)$ of (5) representing the actual surface wave potential must also be a linear combination

$$w(x) = \begin{cases} \beta_0^+ u_0^+(x) + \beta_0^- u_0^-(x) & \text{on} \quad D_0 \quad (14) \\ \beta_1^+ u_1^+(x) + \beta_1^- u_1^-(x) & \text{on} \quad D_1 \quad (15) \end{cases}$$

and since D_0 and D_1 overlap, (11) and (13) imply

$$\beta_0^+ = \mu' e^{i\pi/6} \beta_1^+ \{1 + O(\epsilon)\},$$
$$\beta_0^- = \mu\, e^{i\pi/6}[\beta_1^- - i\beta_1^+ \{1 + O(\epsilon)\}]. \tag{16}$$

These establish a direct, asymptotic connection between the near-shore representation (14) and the deep-sea representation (15) of the wave potential and thereby express, to the first approximation, all the information needed from the differential equation for the scattering problem.

4. Shore absorption

The ratio

$$R_A = \beta_0^+/\beta_0^- \tag{17}$$

in (14) is the shore reflection coefficient expressing the (complex) amplitude β_0^+ of the wave u_0^+ travelling out from shore in terms of that of the wave u_0^- travelling toward shore.

The domain D_1 in (15) contains the image of a segment of the real x-axis beyond X (Fig. 1); in fact, the segment (X, ∞) is mapped on the negative real ξ_1-axis (Fig. 2) for real η. Since $f(x)$ is analytic on a neighborhood of $(0, \infty)$, D_1 may be extended indefinitely in that direction, consistently with the asymptotic requirements [12]. This remains true for sufficiently near-real η, moreover, by the hypothesis of a sectorial domain of analyticity for $h(x)$. D_1 then contains a direction of indefinitely decreasing *re* ξ_1 that takes us out to the open sea, and the physical meaning of the wave potential requires that w then remain bounded. It follows from (10) and (15) that $\beta_1^- = 0$. This is the radiation condition.

For a nontrivial solution, therefore, $\beta_1^+ \neq 0$ in (15), and now (16) and (12) transform (17) into

$$\begin{aligned} R_A &= i(\mu'/\mu)\{1 + O(\epsilon)\} \\ &= i \exp[-2\xi_c(\eta)/\epsilon]\{1 + O(\epsilon)\} \end{aligned} \tag{18}$$

which is the first asymptotic approximation to the characteristic relation between the reflection coefficient R_A and the scaled, square frequency $\eta = \epsilon\omega^2$.

The remaining task is to solve it for ω in terms of R_A. Now, ϵ represents the general scale of the seabed slope, which can be normalized so that $\xi_c(1) = (m - \frac{1}{2})\, i\pi\epsilon$, that is,

$$\int_0^X |n^2\epsilon^2 - k^2|^{\frac{1}{2}}\, dx = (m - \tfrac{1}{2})\,\pi\epsilon \tag{19}$$

with sufficiently large integer m. This is just the spectral condition of Shen et al. [3] who have shown it to have precisely one solution ϵ_{mn} for given, not too small integers m and n below a (large) cut-off value dependent on the number n of wave crests counted across the channel. The integer m is a measure of the number of wave crests counted from shore outward in the x-direction.

To distinguish the information on amplitude and phase in the reflection coefficient, it may be written

$$R_A = \exp[\lambda + i\delta - i\pi/2] \tag{20}$$

with real λ, δ so that $\lambda = \log|R_A|$ is the logarithmic decrement of real amplitude in shore reflection and $\delta - \pi/2$, the phase shift. For perfect reflection [3], $R_A = -i$, so that δ is the relative phase shift due to absorption. The normalization (19) then brings (18) into the form

$$\xi_c(\eta) - \xi_c(1) = -(\epsilon_{mn}/2)(\lambda + i\delta) + O(\epsilon^2). \tag{21}$$

But, $\xi_c(\eta)$ is independent of ϵ (for fixed $n\epsilon$) and analytic on a neighborhood of $\eta = 1$, by (12), (6) and (1), which also yield

$$\xi_c'(1) = i \int_0^X (k^2 - n^2\epsilon^2)^{-\frac{1}{2}} \frac{k^2 dx}{1 + (k^2 - 1)h} = i\,|\xi_c'(1)| \neq 0 \tag{22}$$

by (8). The function $\xi_c(\eta)$ therefore maps the ϵ-independent neighborhood $N(1)$ one-one onto an also ϵ-independent neighborhood N' of $\xi_c(1)$. Given $|\lambda + i\delta| = \lambda_1 > 0$, an $\epsilon_0 > 0$ can accordingly be found so that

$$N' \supset \{\xi_c : |\xi_c - \xi_c(1)| < \epsilon_0\lambda_1/2\} = N_1'$$

and the inverse of $\xi_c(\eta)$ maps N_1' back onto a neighborhood $N_1(1) \subset N$ with simple closed boundary on which

$$|\xi_c(\eta) - \xi_c(1)| = \epsilon_0 \lambda_1/2 > \epsilon_{mn} |\lambda + i\delta|/2$$

for all sufficiently small $\epsilon_{mn} > 0$, i.e., for all sufficiently large m, by (19). It follows from the principle of the argument (Rouché's theorem) that (21) then has just one root in $N_1(1)$ and that this root is simple. Moreover

$$\xi_c(\eta) - \xi_c(1) = (\eta - 1)\xi_c'(1) + O[(\eta - 1)^2]$$

for this root, so that it is

$$\eta \sim 1 - \tfrac{1}{2}\epsilon_{mn}(\lambda + i\delta)/\xi_c'(1) + O(\epsilon^2) \tag{23}$$

and has

$$Im\ \eta \sim \tfrac{1}{2}\epsilon_{mn}\lambda/|\xi_c'(1)| + O(\epsilon^2).$$

By (20) and since $\eta = \epsilon_{mn}\omega^2$, the logarithmic time decrement of the trapped wave modes due to shore absorption is therefore

$$Im\ \omega \sim (4\omega|\xi_c'(1)|)^{-1} \log |R_A| \tag{24}$$

where $|\xi_c'(1)|$ is given by (22) and (1), the frequency $\omega = \epsilon_{mn}^{-\frac{1}{2}}$ is given by (19) and (1) with X denoting the root of (6) for $\eta = 1$, and R_A is the amplitude reflection coefficient at the shore.

It should be mentioned that the mathematical argument is complete in deducing (24) from the refraction equation (2) as the first asymptotic approximation to the decay rate only on the implicit assumption that $|R_A|$ is independent of ϵ. If $|R_A| \to 1$ as $\epsilon \to 0$, then standard turning point theory may be inadequate for determining the decay rate and a much more subtle analysis on the lines of [6] may be necessary, but in the present state of knowledge on R_A, this would appear premature.

5. Shore absorption for round islands

For axisymmetric seabed topography it is convenient to employ polar coordinates r, θ in the mean water surface, with r denoting distance from the island center, normalized so that the island radius is 1. Then h depends only on r, with $h(1) = 0$ and $h'(1) > 0$ for a typical shore, and will again be assumed to increase monotonely with r and to be an analytic function of r on a (sectorial) neighborhood of the real r-axis.

From the dispersion relation (1), the wave number function $k = k(r)$ is then also seen to be analytic on a neighborhood of $(1, \infty)$ with singularity at $r = 1$, limit $k_\infty \geqslant \eta = \epsilon\omega^2$ as $r \to \infty$, and monotone decrease in r between, for real r and η.

The potential may now be Fourier-analyzed with respect to the angular coordinate θ in terms of modes

$$\Psi(r, \theta) = e^{in\theta} r^{-\frac{1}{2}} w(r) \tag{27}$$

for which the refraction equation (4) reduces to

$$d^2 w/dr^2 = [\epsilon^{-2} f(r) + g''/g]\, w(r), \tag{28}$$

$$f(r) = (n\epsilon/r)^2 - k^2, \tag{29}$$

$$g(r) = (rG)^{\frac{1}{2}}. \tag{30}$$

This looks quite similar to (5), (6), and $g(r)$ has the same type of branch point at the shore as G, but $f(r)$ has a character quite different from that of the monotone functions $f(x)$ in (6), and this makes the round island much more typical of natural topographies.

Since $k \to k_\infty \geqslant \eta$ as $r \to \infty$ (for real η), $rk(r) \to \infty$ both as $r \to \infty$ in the open sea and as $r \to 1$ at the shore (where k is singular), and therefore $rk(r)$ has a minimum. For simplicity, extreme shelves will again be excluded, and then $rk(r)$ has only one minimum in $[1, \infty]$ for real η [3]. This minimum is a cut-off value for $n\epsilon$ below which no trapping occurs [3] and it will be assumed here that $n\epsilon$ takes a fixed value substantially above this cut-off value. Then $f(r)$ in (29) has two roots, r_1 and $r_2 > r_1$, in $(1, \infty)$ for real η, and (28) therefore has two turning points r_1, r_2 in addition to the singular turning point at the shore $r = 1$. Between r_1 and r_2, $f(r) > 0$ and the solution of (28) have there the same monotone, non-oscillatory character as have those of (5) for $x > X$, where $f(x) > 0$. But in contrast to the case of channel topographies, this solution character does not extend to the open sea: for $r > r_2$, $f(r) < 0$ and the solutions of (28) resume the wave character which they have between the shore and $r = r_1$.

The roots r_1, r_2 of $f(r)$ depend on $\eta = \epsilon\omega^2$, by (29) and (1), and for complex η, $r_1(\eta)$ and $r_2(\eta)$ are also complex. For near-real η, however, they remain near the real r-axis, as indicated in Fig. 3, which also shows the Stokes Lines from the turning points, on which

$$\int^r [f(\rho)]^{\frac{1}{2}}\, d\rho$$

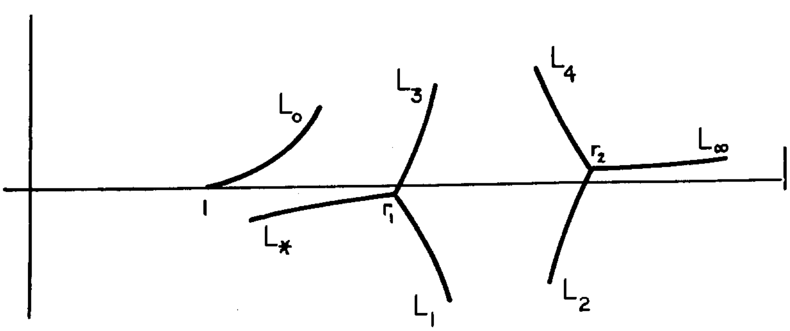

Fig. 3.

has constant real part. For real η, L_0 and L_* again coincide with the segment $(1, r_1)$ of the real r-axis, and a Stokes Line L_∞ coincides with (r_2, ∞) and therefore leads out to the open sea.

The analysis of (28) proceeds [6] along lines similar to Section 3. The natural variables are

$$\xi_i(r_j, r) = \int_{rj}^{r} [f_i(\rho)]^{\frac{1}{2}}\, d\rho \tag{31}$$

where r_j denotes a turning point ($j = 0$, 1 or 2 with $r_0 = 1$) and $f_i^{\frac{1}{2}}$, the branch of $f(r)^{\frac{1}{2}}$ that makes $\arg \xi_i = \pi/2$ and $|\xi_i|$ increasing with $|r - r_j|$ along the Stokes line L_i ($i = 0, *, 1, \ldots 4$ or ∞, cf. Fig. 3); the choices may be made in close analogy with (8), (9). Each Stokes line is then again associated with a pair of WKB solutions $u_i^+(r)$, $u_i^-(r)$ such that u_i^+ represent a wave travelling away from the turning point r_j along L_i and u_i^-, a wave incident on the turning point along L_i. These are exact and independent solutions, and the solution $w(r)$ of (28) representing the actual potential must again be

$$w(r) = \begin{cases} \beta_0^+ u_0^+(r) + \beta_0^- u_0^-(r) \\ \\ \beta_\infty^+ u_\infty^+(r) + \beta_\infty^- u_\infty^-(r) \end{cases} \tag{32}$$

with coefficients β_i depedent on η, but not on r. Those coefficients must be related, in turn, and standard theory [12] yields asymptotic appoximations to their relations, in particular [6],

$$\gamma_0 \beta_\infty^- = \beta_0^+ \{i + O(\epsilon)\} + \beta_0^- \exp\,[-2\xi_0(1, r_1)/\epsilon]\,\{1 + O(\epsilon)\} \tag{33}$$

with γ_0 bounded as $\epsilon \to 0$ and $\xi_0(1, r_1)$ independent of ϵ for fixed $n\epsilon$.

From (32), the ratio

$$R_A = \beta_0^+/\beta_0^- \tag{34}$$

is again seen to be the (complex) amplitude reflection coefficient. The radiation condition for islands, however, is quite different from that for channel topographies (Section 4) because the far field, beyond $r = r_2$, is now a wave field. For real η, where L_∞ coincides with (r_2, ∞), u_∞^+ is a pure wave radiated out to sea and u_∞^-, a pure wave incident from the open sea. For near-real η, this remains essentially true at large real r, even though the waves now have some degree of growth or decay with increasing r [6]. It follows from (32) that any solution $w(r) \not\equiv 0$ must involve radiation out to, or in from, the open sea, or both. The existence question for trapped wave modes is therefore the following: Can (28) have nontrivial solutions $w(r)$ without energy supply by radiation incident from the open sea? That is, for what values of η are there solutions with incident amplitude coefficient

$$\beta_\infty^- = 0$$

in (32), but with $\beta_\infty^+ \neq 0$?

The answer is supplied directly by (33), (34), namely

$$0 = R_A \{i + O(\epsilon)\} + \exp[-2\xi_0(1, r_1)/\epsilon]\{1 + O(\epsilon^2)\} \tag{35}$$

to a first approximation for small ϵ. For $\eta = 1$,

$$\xi_0(1, r_1)\Big|_{\eta=1} = i\int_1^r |k^2r^2 - n^2\epsilon^2|^{\frac{1}{2}} r^{-1}\, dr = \xi_c(1) \tag{36}$$

from (31), (29), but for $\eta \neq 1$, $\xi_0(1, r_1) = \xi_c(\eta)$, by (31), (29) and (1). Now ϵ can again be normalized so that

$$\xi_c(1) = (m - \tfrac{1}{2})i\pi\epsilon$$

with sufficiently large integer m, which is the spectral condition of Shen et al. [3] for axisymmetric topographies and has precisely one solution ϵ_{mn} for given, not too small, integers m and n (below a large cut-off value) related to the crest count as in Section 4. With R_A again expressed in the form (20) the characteristic condition (35) becomes (21), whence we may proceed literally as in Section 4 to obtain (24). The only change is that now [6]

$$|\xi_c'(1)| = \int_1^{r_1} |k^2 - (n\epsilon/r)^2|^{\frac{1}{2}} k^2 [1 + (k^2 - 1)h]^{-1}\, dr \tag{37}$$

by (36), (31), (29) and (1).

The asymptotic argument just summarized depends again on the implicit assumption that the reflection coefficient $R_A \not\to 1$ as $\epsilon \to 0$, which appears plausible. The theory, however, involves two basic limits, viz., wave amplitude $\to 0$ and seabed slope $\epsilon \to 0$, and there is some evidence [14] that the limit of R_A may depend on the order in which the basic limits are taken! If $R_A \to 1$, moreover, (24) predicts zero decay rate in the limit $\epsilon \to 0$, which is hard to reconcile with the prediction of (32) that a nontrivial solution without energy supply by radiation incident from the open sea must necessarily experience an energy loss by radiation out to the sea. It would be necessary to conclude that the solution tends to a trivial one as $\epsilon \to 0$ and therefore, that the scaling underlying the formulation of the problem in Section 2 must be at a fault?

For such reasons, Lozano and Meyer [6] made a much more precise analysis of (28) based on symmetries in the complex plane of $\eta = \epsilon\omega^2$ closely related to the property of energy conservation inherited by (28) via the refraction equation (2) from the classical surface wave equations. They were able to prove that the connection formula (33) between the coefficients in (32) is actually of the form

$$\gamma_0\beta_\infty^- = \beta_0^+[1 - (1 - i)\gamma^{-2}\{1 + O(\epsilon)\}] + \beta_0^- \exp[-\tfrac{2}{\epsilon}\xi_c + i\epsilon\sigma - \frac{i\pi}{2}] \tag{38}$$

with $\sigma(\eta)$ bounded on the neighborhood N of $\eta = 1$, and real for real η, and

$$\gamma = \exp[\epsilon^{-1} \int_{r_1}^{r_2} |n^2\epsilon^2 - k^2r^2|^{\frac{1}{2}} r^{-1}\, dr]. \tag{39}$$

The integral in (39) is positive for real η so that γ^{-2} in (38) is actually smaller than any power of ϵ. If we now normalize ϵ so that

$$\xi_c(1) = (m - \tfrac{1}{2})\, i\pi\epsilon + \tfrac{1}{2}\, i(\sigma - \epsilon\delta),$$

then (34) and (20) express the radiation condition $\beta_\infty^- = 0$ as

$$\xi_c(\eta) - \xi_c(1) = -\tfrac{1}{2}\, \epsilon_{mn}\, \lambda + (1 - i)\, \gamma^{-2}\, \{1 + O(\epsilon)\} \tag{40}$$

which has a far, far smaller error term than (21) and implies again (24), provided only that $\lambda = \log |R_A| \to 0$ with ϵ more slowly than γ^{-2}. This removes the weakness of the WKB analysis first outlined for the derivation of (24); that formula is now seen to give the decay rate even if the reflection coefficient $R_A \to 1$ quite fast as the seabed slope $\epsilon \to 0$.

Acknowledgments

This work was supported in part by the National Science Foundation under Grant MCS 77-00097 and by the Wisconsin Alumni Research Foundation. Figures 2, 3 are from the Physics of Fluids [6].

REFERENCES

[1] F. Ursell, Edge waves on a sloping beach, *Proc. Roy. Soc. London* A 214 (1952) 79-97.
[2] M. S. Longuet-Higgins, On the trapping of wave energy around islands, *J. Fluid Mech.* 29 (1967) 781-821.
[3] M. C. Shen, R. E. Meyer and J. B. Keller, Spectra of water waves in channels and around islands, *Phys. Fluids* 11 (1968) 2289-2304.
[4] J. B. Keller, Surface waves on water of non-uniform depth, *J. Fluid Mech.* 4 (1958) 607-614.
[5] J. C. W. Berkhoff, Computation of combined refraction-diffraction. *Proc. 13th Confer. Coastal Engrg.* edited by J. W. Johnson, pp. 471-490, Amer. Soc. Civil Engrs, New York (1973).
[6] C. Lozano and R. E. Meyer, Leakage and response of waves trapped by round islands, *Phys. Fluids* 19 (1976) 1075-1088.
[7] J. Harband, Propagation of long waves over water of slowly varying depth, *J. Engrg. Math.* 11 (1977) 97-119.
[8] W. Summerfield, Circular islands as resonators of long-wave energy, *Phil. Trans. Roy. Soc. London* A 272 (1972) 361-402.
[9] J. J. Stoker, *Water waves*, Wiley (Interscience), New York (1957).
[10] J. Heading, *An introduction to phase-integral methods*, Wiley, New York (1962).
[11] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley (Interscience), New York (1965).
[12] M. A. Evgrafov and M. V. Fedoryuk, Asymptotic behaviour as $\lambda \to \infty$ of the solution of the equation $w''(z) - p(z, \lambda)\, w(z) = 0$ in the complex z-plane, *Usp. Mat. Nauk* 21 (1966) 3-50 [Russ. Math. Surv. 21 (1966) 1-48].
[13] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York (1974).
[14] R. E. Meyer and A. D. Taylor, Run-up on beaches, in *Waves on beaches*, R. E. Meyer, ed., Academic Press, New York (1972).